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Noether's Theorem in Symplectic Bundles

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After an analysis of some types of symplectic bundles, we use seeded fiber bundles to find laws of conservation in relativity from different dynamical groups. By using the concept of dynamical group in a bundle, we obtain an extension of Noether's theorem to fiber bundle structures.

1. INTRODUCTION

A usual practice in relativity is to study new metrics satisfying Einstein's equations even if they do not have a direct connection with the action of Poincaré or Lorentz groups on the space-time manifold where those metrics are defined. In fact, there is a tendency to use the conformal group (Kea, 1997) in order to obtain equivalent conformal classes, and hence a simpler description of the physical system.

In our opinion, the relationship between conservation laws and dynamical groups has not been sufficiently studied; it is usual to consider the same laws of conservation even if a group different from the Lorentz or Poincaré group is used. Our purpose is to investigate whether this makes sense. We start from Souriau's symplectic formulation of Noether's theorem (Souriau, 1997). This formulation is described by presymplectic manifolds whose quotient by the characteristic foliation is the symplectic manifold or dynamic system that we want to study. *If V is a presymplectic manifold with 2-form of Lagrange σ, and μ is a moment of a dynamic group over V, then μ is constant in every leaf of the characteristic foliation* ker σ.

In previous work (Liern and Olivert, 1995a, b, 1999) we proved that the use of certain symplectic fibrations (seeded fiber bundles), in which a

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connection and an horizontal foliation exist, allows us to generalize some results of symplectic mechanics to general relativity; therefore it is reasonable to give a description of Noether's theorem in this structure. Two main reasons justify working with a seeded fiber bundle structure: (i) to unify relativity and symplectic mechanics in the same structure and (ii) to plan possible replacements of the dynamical group (structural group of fiber bundle).

Clearly, dealing with structures more general than symplectic manifolds allows us to describe the evolution of dynamical systems in a more precise way (Grigore, 1992; Gotay *et al.*, 1983; Liern and Olivert, 1999). However a detailed study to guarantee the extension of the properties must be performed.

2. WHY WE USE SYMPLECTIC BUNDLES IN RELATIVITY

Since gauge theory was successfully explained in terms of principal bundles with a connection (Wu and Yang, 1975) the use of fiber bundles to explain physical phenomena has notably grown. More precisely, effort in symplectic mechanics has aimed at constructing associated fiber bundles with a symplectic manifold as a fiber representing the analyzed dynamical system. The existence of a connection in these fiber bundles is fundamental because this connection reflects the underlying interaction. Essentially two types of mathematical constructions exist depending on the geometrical structure imposed on the dynamical system.

Definition 1. Let $\lambda = (M, \pi, B, E)$ be a vector bundle with base space B and a symplectic vector space E as fiber type. λ is called a symplectic vector bundle if there exists a 2-form Ω such that for every $p \in B$, Ω_p is a symplectic form in E_p.

These bundles are useful for studying Lagrangian submanifolds associated with the concept of polarization in geometric quantization (Morvan, 1984), but they do not permit us to study many results of symplectic mechanics that can not be represented in terms of vector spaces. For example, from Souriau's viewpoint, elementary particles with spin can be described with the symplectic manifold $\mathbb{R}^6 \times S^2$ if the particle has mass or with $\mathbb{R}^4 \times S^2$ if the particle is massless (Souriau, 1970, 1997). Therefore, a vector bundle cannot describe these dynamical systems.

This question can be resolved by using a more general symplectic fibration structure due to Guillemin *et al.* (1996):

Definition 2. A fibration (M, π, B, U) over B with fiber type U is a simplectic fibration if the following conditions hold:

(i) The fibers U_b are all symplectic manifolds.

(ii) There exists a 2-form w on M that satisfies $i(v_1 \wedge v_2)\, dw = 0$ for every pair of vertical vector fields v_1 and v_2.

(iii) The restriction of w to each fiber, i.e., $w|_{U_b}$, is the symplectic form of the fiber.

Remark 1. In a symplectic fibration there exists a natural (Guillemin *et al.*, 1996) symplectic connection Γ compatible with the 2-form w. Moreover, the parallel displacements of Γ are symplectomorphisms.

This structure allows us to reformulate symplectic mechanics in terms of fiber bundles, but it does not consider explicitly many fundamental aspects. These fibrations are not associated to any principal bundle and, moreover, there is not a structural group. This makes it difficult to use it in a dynamical group theory. Besides, we know (Souriau, 1997) that the evolution of a dynamical system (U, σ) whose evolution space is a presymplectic manifold (V, σ_V) can be obtained by studying the characteristic foliation of V. Definition 2 does not assume the existence of presymplectic manifolds (and therefore the existence of foliations) inherent in the structure itself. In our opinion, this structure is too general to be operative.

We propose to work with a different structure: the *seeded fiber bundles*. It was introduced in Liern and Olivert (1995a) and we will prove that the original definition is equivalent to Definition 2 plus the following three conditions:

C1. The fibration (M, P, B, U) is a G-bundle associated to a principal G-bundle with connection.

C2. The fiber bundle contains a foliation that shows the evolution of the dynamical system.

C3. The total space of the bundle contains presymplectic manifolds, which are evolution spaces of the dynamical systems.

In particular, if we consider such a fiber bundle over a space-time manifold $\mathcal{M}$ and the Poincaré group as structural group, we can obtain, for instance, the main results concerning free elementary particles in general relativity (Liern and Olivert, 1995a, 1999).

The original definition of a seeded fiber bundle is the following (Liern and Olivert, 1995b):

Definition 3. Let $\lambda = (P, B, \pi, G)$ be a principal G-bundle with a connection $\mathcal{H}$ and let (F, σ) be a G-space Hausdorff symplectic manifold such that G acts on F by symplectomorphism. A fiber bundle $\lambda[\mathrm{F}] = (P_F, B, \pi_F, G)$ associated with λ and with fiber F is called a seeded fiber bundle (SFB) if there exists a projectable (Brickell and Clark, 1970) foliation S contained in the horizontal distribution $\mathcal{Q}$ induced by $\mathcal{H}$ such that every fiber cuts each leaf of S at most in one point.

We will denote these symplectic bundles by $\lambda[F](S) = (P_F, B, \pi_F, G; S)$ in order to make the foliation S appear explicitly.

In Definition 3 we require that G acts on F by symplectomorphisms to preserve the symplectic structure of fiber type. Actually, we have the following result:

Proposition 1. Let $\lambda[F](S)$ be a seeded fiber bundle. There exists a natural simplectic structure over each fiber.

Proof. Let $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ be local trivializations for δ defined by $\varphi_\alpha(p) = (\pi(p), \psi(p))$. We can define local trivializations for $\lambda[F]$, $\tilde{\varphi}_\alpha$: $\pi_F^{-1}(U_\alpha) \to U_\alpha \times F$, given by

$$\tilde{\varphi}_\alpha((p, f)G) = (\pi(p), \psi_\alpha(p) \cdot f) \tag{1}$$

We have $\tilde{\varphi}_\alpha(b) = pr_2 \circ \tilde{\varphi}_\alpha|_{F_b}\colon F_b \to F$, where $pr_2\colon U_\alpha \times F \to F$ is the projection to the second factor. We obtain the maps $\tilde{\varphi}_{\beta\alpha}(b) = \tilde{\varphi}_\beta(b) \circ \tilde{\varphi}_\alpha(b)^{-1}\colon F \to F$ given by

$$\tilde{\varphi}_{\beta\alpha}(b)\,(f) = g_{\beta\alpha}(b) \cdot f$$

where $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to G$ are transition functions of δ. As G acts by symplectomorphism on F, the maps $\tilde{\varphi}_{\beta\alpha}(b)$ are symplectomorphisms for each b in $U_\alpha \cap U_\beta$. We can define a symplectic form σ_b of F_b by

$$\sigma_b = \tilde{\varphi}_\alpha\,(b)^*\,\sigma \tag{2}$$

This structure is natural because it is independent of the local diffeomorphism $\tilde{\varphi}_\alpha$. ■

Proposition 1 guarantees that the SFB structure satisfies conditions (i) and (iii) of Definition 2. Condition (ii) will be considered below. Moreover, Definition 3 clearly satisfies conditions C1 and C2, too, but we need a presymplectic structure to prove that condition C3 holds. In order to relate the presymplectic manifolds with the SFB structure, we give the following characterization (Liern and Olivert, 1995a):

Theorem 1. Let $\lambda = (P, B, \pi, G)$ be a principal G-bundle with a connection $\mathcal{H}$ and let (F, σ_F) be a symplectic Hausdorff manifold left G space. A fiber bundle $\lambda[F] = (P_F, B, \pi_F, G)$ associated to λ with fiber type F, is SFB if and only if for every $m \in B$ there exists a presymplectic regular manifold $(V_m, \sigma_m) \subset P_F$ satisfying the following:

(a) The family $E := \{V_b | b \in B\}$ covers all P_F.
(b) dim $V_m = k$ (constant) $\forall m \in B$.
(c) There is a surjective submersion $\psi_m\colon V_m \to \pi_F^{-1}(m)$ such that $\pi_F^{-1}(m) = V_m/\ker \psi_{m*}$.
(d) Given $m, n \in B$, if $V_m \cap V_n \neq \emptyset$, then $V_m = V_n$.

(e) $\ker(\sigma_m)_w \subset \mathcal{Q}_w$, where $\mathcal{Q}$ is the horizontal distribution in P_F induced by $\mathcal{H}$, and $w \in V_m$.

This theorem also allows us to see an SFB, $\lambda[F](S) = (P_F, B, \pi_F, G; S)$, as a bundle of fibrations in which every fiber $\pi_F^{-1}(m)$, $m \in B$, is the base manifold of a new fibration $\lambda_m = (V_m, \pi_F^{-1}(m), \psi_m)$ whose total space is a presymplectic manifold.

This fact, together with (d) in Theorem 1, implies that P_F admits a new differentiable structure determined by the family $E = \{V_b | b \in B\}$. Although P_F is not usually a presymplectic manifold, E admits a presymplectic structure. Let $z \in E$; by (d) in Theorem 1, a unique presymplectic manifold V_m such that $z \in V_m$ exists. The presymplectic 2-form in E is given by

$$\sigma_E(X_z, Y_z) = \sigma_{V_m}(X'_z, Y'_z) \tag{3}$$

where X' (resp. Y') is a vector field i-related to X (resp. Y), and $i: V_m \to E$ is the canonical immersion.

3. SEEDED FIBER BUNDLES OVER SPACE-TIME MANIFOLDS

Let $\lambda = (P, \mathcal{M}, \pi, G)$ be a principal bundle with connection, over a space-time manifold $\mathcal{M}$, and with an arbitrary structural group G. Let (U, σ) be a symplectic Haussdorf left G space that represents a dynamical system (Kea, 1997). We can construct a seeded fiber bundle over $\mathcal{M}$ (Liern and Olivert, 1995a, b)

$$\lambda[U](S) = (P_U, \mathcal{M}, \pi_U, G; S) \tag{4}$$

The foliation S can be projected on a foliation Ω over $\mathcal{M}$,

$$\Omega := \{X \mid X \text{ is a vector field in } B\text{: } \pi_{U*}\, Y = X \circ \pi_U,\ Y \in S\} \tag{5}$$

such that $\dim \Omega = \dim S$; because of this we can study the evolution of the dynamical system U over $\mathcal{M}$ from the integral manifolds of the foliation Ω.

Of course, not all foliations that make $\lambda[U]$ to be a seeded fiber bundle can describe the evolution of the dynamical system (U, σ). We have to impose an additional condition on the foliation. Let X be a vector field of Ω, c a maximal integral curve of X, and $\tau_t^c\colon T_{c(0)}\,\mathcal{M} \to T_{c(m)}\mathcal{M}$ the parallel displacement along $c(t)$ $\forall t \in I$, where I is the domain of c. We say that $\lambda[U](S)$ is provided with a *motion law* if

$$\tau_t^c\, \Omega(c(0)) = \Omega(c(t)), \qquad t \in I \tag{6}$$

With the fiber bundle (4) and the motion law (6) we extend free elementary particles results (Souriau, 1970, 1997) of symplectic mechanics to general relativity (Liern and Olivert, 1995a, 1999). The idea for this extension is that depending on the dimension of Ω (one or three), the motion law originates geodesics or totally geodesic integral manifolds in $\mathcal{M}$, respectively.

However, the motion law described by (6) does not allow us to work with symplectomorphisms because the space-time manifold is neither presymplectic nor symplectic. In consequence, we cannot study physical magnitudes that are preserved through time.

According to Theorem 1, the total space P_F contains some presymplectic manifolds that "cover" the fibers. Then, each $m \in \mathcal{M}$ in the SFB contains a dynamical system, $\pi_F^{-1}(m)$, and its evolution space V_m.

As the observation is made in the space-time $\mathcal{M}$, we can impose a condition (which we call the *stability condition*) by using symplectomorphisms that preserve the symplectic and presymplectic structures given in Theorem 1. Let X be a vector field of Ω, c: $I \to \mathcal{M}(I \subset \mathbb{R})$ an integral curve of X, and τ_t^c: $\pi_U^{-}(c(0)) \to \pi_U^{-1}(c(t))$ the parallel displacement along c in P_U. Given $p \in P_U$, we can define the curve $\beta^c(t, p)$: $I \to P_U$ as

$$\beta^c(t, p) = \tau_t^c(p), \quad t \in I, \qquad \text{where} \quad \pi_U(p) = c(0)$$

For every $p \in \pi_U^{-1}(c(0))$, we have that $\{\beta^c(t, p)\}_{p \in \pi_U^{-1}(c(0))}$ is a family of differentiable curves in P_U.

The foliation S is contained in the horizontal distribution. Because of this, we have that $(\partial/\partial t)\beta^c(t, p) \subset S(\beta^c(t, p))$; then $\{\beta^c(t, p)\}_{p \in \pi_U^{-1}(c(0))}$ induces a family of curves

$$\gamma^c(t, p): \quad I \times E \to E \tag{25}$$

where E is the manifold given by (a) in Theorem 1.

Definition 4. We say that the SFB $\lambda[U](S)$ is *c-stable* if for every $t_0 \in I$, $\gamma^c(t_0, p)$ is a symplectomorphism.

Remark 2. A sufficient condition for the c-stability is that the presymplectic manifolds in Theorem 1 are symplectomorphics (Liern and Olivert 1999).

If $\lambda[U](S)$ is c-stable, there exist some curves in P_U that preserve the foliation in the presymplectic manifolds in such a way that the fibers coincide with the parallel displacement. Besides, these displacements are symplectomorphisms. Hence, we obtain Remark 1.

The c-stability condition expressed in Definition 4 is inherited via pullback. Given $\lambda[F](S) = (P_F, \pi_F, \mathcal{M}, G; S)$ a SFB c-stable, L a submanifold of $\mathcal{M}$ and i: $L \to \mathcal{M}$ the canonical immersion. If the pullback bundle $i^*(\lambda[F]) = (L \times_{\mathcal{M}} P_F, i^* \pi_F, L, G)$ is a seeded fiber bundle, then $i^*(\lambda[U])(S^*)$ is $\tilde{c}$-stable, where $c = i \circ \tilde{c}$.

4. KILLING FIELDS AND NOETHER'S THEOREM

In order to study Noether's theorem in fiber bundles, we need to extend two fundamental concepts in this theorem: the Killing fields and dynamical groups. We make use of Souriau's Killing field formulation (Souriau, 1997):

Definition 5. Let G be a Lie group, $\mathcal{M}$ a manifold, and $\phi_g\colon \mathcal{M} \to \mathcal{M}$, $\forall_g \in G$, the G-action on $\mathcal{M}$. A Killing field is a map $Z_{\mathcal{M}}\colon \mathcal{M} \to T\mathcal{M}$ given by $Z_{\mathcal{M}}(m) = \phi_{m*e}Z$, where $Z \in T_eG$.

In particular, if there is a metric tensor $\mathbf{g}$ in $\mathcal{M}$ and G acts by isometries, Definition 5 is equivalent to the usual one: "X is a Killing field if $Ł_X\mathbf{g} = 0$," where Ł denotes the Lie derivative.

In seeded fiber bundles with a structural dynamical group G acting on the fiber, we cannot assert anything about the action on presympletic manifolds that appear in Theorem 1. To study this situation, we need to extend the concept of dynamical group to fiber bundles:

Definition 6. Let $\lambda[F](S) = (P_F, \pi_F, B, G; S)$ be an SFB such that G is a dynamical group acting by symplectomorphisms over the fiber type F. The group G is called a dynamical group over $\lambda[F](S)$ if for every curve c of B the diagram

$$\begin{array}{ccc} \pi^{-1}(c(t_1)) & \xrightarrow{\tau_{t_1c}^{t_2}} & \pi^{-1}(c(t_2)) \\ g\big\downarrow & & g\big\downarrow \\ \pi^{-1}(c(t_1)) & \xrightarrow{\tau_{t_1c}^{t_2}} & \pi^{-1}(c(t_2)) \end{array}$$

is commutative for all $g \in G$.

Now, we can deal with a generalization of Noether's theorem to the symplectic fibration over a Riemannian manifold.

Let $\lambda[TV](S) = (P_{TV}, V, \pi_{TV}, G; S)$ be a seeded fiber bundle over a Riemannian manifold (V, g) with fiber type TV and horizontal foliation S. If $c\colon I \to V$ is a temporal geodesic in V, then Im c is a submanifold. We can define

$$\mathcal{G}(m) := \Omega(m) \cap i_*T_m(\mathrm{Im}\, c), \qquad \forall m \in \mathrm{Im}\, c$$

where Ω is the projected foliation of S over V.

As c and Ω are temporal (or light-like), we obtain that $\Omega(m) \cap i_*T_m(\mathrm{Im}\, c) \neq \emptyset$, $\forall m \in \mathrm{Im}\, c$, and as $\dim \mathrm{Im}\, c = 1$, we have that $\dim \Omega_1(m) = 1$, $\forall m \in \mathrm{Im}\, c$. Then the pullback $i^*(\lambda[TV](S))$ given by

$$\begin{array}{ccc} i^*(P_{TV}) & \xrightarrow{pr_2} & P_{TV} \\ pr_1\big\downarrow & & \pi_{TV}\big\downarrow \\ \mathrm{Im}\, c & \xrightarrow{i} & B \end{array}$$

is a seeded fiber bundle.

Assuming that $\lambda[TV](S)$ is c-stable, the parallel displacements in $i^*(\lambda[TV](S))$ are symplectomorphisms.

Proposition 2. If G is a dynamical group over $i^*(\lambda[TV](S))$, then G is a dynamical group over every presymplectic manifold and has a moment.

Proof. Let V'_m be a presymplectic manifold of $i^*(P_{TV})$ and let μ_m: $V'_m \to \pi_{TV}^{-1}(m)$ be the canonic submersion. In a seeded fiber bundle, every fiber is submanifold of its presymplectic manifold. Let i_m: $\pi_{TV}^{-1}(m) \to V_m$ be a map such that $\mu_m \circ i_m = 1_{\pi_{TV}^{-1}(m)}$. We can define an action of G over V_m, ϕ_g: $V_m \to V_m$, $\forall g \in G$, as

$$\phi_g(p) := i_{c(m)}\tau_c g \tau_c^{-1}\mu_{c(m)}(p), \qquad \forall m \in V_m$$

It is an action because

$$\begin{aligned}\phi_g\phi_h(p) &= \phi_g(i_{c(m)}\tau_c g\tau_c^{-1}\mu_{c(m)}(p))\\ &= i_{c(m)}\tau_c g\tau_c^{-1}\mu_{c(m)}i_{c(m)}\ \tau_c h\tau_c^{-1}\mu_{c(m)}(p)\end{aligned}$$

This action induces over the fiber the initial action

$$\begin{aligned}\psi_g\mu_{c(m)}(p) &= \mu_{c(m)}\phi_g(p) = \mu_{c(m)}i_{c(m)}\ \tau_c g\tau_c^{-1}\mu_{c(m)}(p)\\ &= \tau_c g\tau_c^{-1}\mu_{c(m)}(p) = g\mu_{c(m)}(p)\end{aligned}$$

where ψ_g is the action over U induced by the action over V.

Let us see that it is symplectic. The presymplectic form is $\sigma_{V_m} = \mu^*\sigma$, and therefore

$$\phi_g^*\ \sigma_{V_m} = \mu_{c(m)}^*\ \tau_c^{-1^*}\ g^*\ \tau_{c(m)}^*\ i_{c(m)}^*\ \sigma_{V_m}$$

As $i_{c(m)}^*\ \mu_{c(m)}^*\ \sigma = \sigma$, we obtain $i_{c(m)}^*\ \sigma_{V_m} = \sigma$, and

$$\phi_g\sigma_V = \mu_{c(m)}^*\ \tau_c^{-1^*}\ g^*\ \tau_{c(m)}^* = \mu_{c(m)}^*\ \sigma = \sigma_{V_m}$$

Therefore G is a dynamic group over V_m.

The Killing fields on V_m and the Killing fields on $\pi_{TV}^{-1}(m)$ are μ-related, i.e.,

$$\mu_*\ Z_{V_m} = Z_{\pi_{TV}^{-1}(m)} \circ \mu$$

However, TV is a potential manifold and admits a moment, and therefore every fiber $\pi_{TV}^{-1}(m)$ admits a moment.

Let J be a moment of $\pi_{TV}^{-1}(m)$; we can define

$$M(Z) := \mu^*\ (\hat{J}(Z))$$

Finally, let us see that M is a moment of the structural group in the manifold V_m:

$$dM(Z) = d\mu^*(\hat{J}(Z)) = \mu^* d\,(\hat{J}(Z)) = -\mu^* i_{Z_{\pi^{-1}(m)}}\,\sigma$$
$$= -\mu^* i_{\mu_* Z_{V_m}}\,\sigma = -i_{Z_{V_m}}\,\mu^*\sigma = -i_{Z_{V_m}}\,\sigma_{V_m} \quad \blacksquare$$

If $\lambda[TV](S)$ is c-stable, we obtain that the parallel displacements in $i^*(\lambda[TV](S))$ are symplectomorphisms.

The horizontal curves are leaves of the characteristic foliation $\sigma_{V'}$ because its tangent spaces coincide with the horizontal 1-distribution and therefore these horizontal lifts are tangent to the horizontal curves.

Let $u_m \in T_mV$; for each u in a horizontal curve that meets the fiber $\pi_{TV}^{-1}(m) = T_mV$ in u_m, we obtain that $\mu(u) = u_m$ because the image of a leaf element under projection μ is the unique element of the intersection of the leaf and the fiber. We have

$$[M(Z)](u) = [\mu^*(\hat{J}\,(Z))](u) = [\hat{J}(Z)]\mu\,(u) = [\hat{J}(Z)](u_m)$$

and by Riemannian manifold theory

$$[\hat{J}(Z)](u_m) = g(Z_{V_m}, u_m)$$

and we have

$$[M(Z)](u) = g(Z_{V_m}, u_m) = \text{const}$$

In our presymplectic manifold, we obtain that the moment is preserved in each leaf and therefore Noether's theorem holds.

5. CONCLUSION

In theoretical physics it is usual to apply Noether's theorem with the same formulation in both symplectic and presymplectic manifolds. In practice these fields of application are understood to be equivalent, but even so, a rigorous description is needed. Studies of this problem have been made using the Poincaré group, but in this paper we generalize it to other groups.

We show that a pullback of a seeded fiber bundle allows us to reproduce symplectic and differential structures in the fiber bundle whose base space is an observer, i.e., a smooth curve. We also prove (Proposition 2) that the structural group of the fiber bundle is dynamic over every presymplectic manifold and has a moment for the group action. Then Noether's theorem can be used in the seeded fiber bundle structure, not only in its fibers (symplectic manifolds), but also in the presymplectic manifolds that cover its fibers.

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